

# Infrared behaviour and fixed points in Landau gauge QCD

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We investigate the infrared behaviour of gluon and ghost propagators in Landau gauge QCD by means of an exact renormalisation group equation. We explain how, in general, the infrared momentum structure of Green functions can be extracted within this approach. An optimisation procedure is devised to remove residual regulator dependences. In Landau gauge QCD this framework is used to determine the infrared leading terms of the propagators. The results support the Kugo-Ojima confinement scenario. Possible extensions are discussed.

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Charged particles have long-range nature, whereas localised objects are neutral. For local quantum field theories one derives more generally that every gauge-invariant localised state is singlet with respect to the unbroken global charges of the gauge symmetry. Thus, in QCD without Higgs mechanism any localised physical state must be colourless. This extends to all physical states only if a mass gap is present. Then, colour-electric charge superselection sectors cannot arise, and every gauge-invariant state must also be a colour singlet. These are the signatures of confinement.

Within covariant linear gauges, the necessary conditions for confinement were formulated more than twenty years ago by Kugo and Ojima [1]. They express the necessity of a mass gap and the conditions for avoiding the Higgs mechanism in a covariant continuum formulation of QCD in terms of local fields, *i.e.*, for the local measurement of (colour) quantum numbers. Both conditions are encoded in the critical infrared exponents of gluonic and ghost correlations in Landau gauge QCD [2]. For the mass gap, the massless transverse gluon states of perturbation theory must be screened nonperturbatively, leading to an infrared suppression for the gluon propagator. In turn, the global gauge charges entail the infrared enhancement of the ghost correlations leading to their infrared dominance.

The infrared behaviour just described has been found in solutions of truncated Dyson-Schwinger equations [3]. Since then, the same qualitative behaviour has emerged in a variety of nonperturbative approaches including an increasing number of lattice simulations [4] and investigations based on Fokker-Planck type diffusion equation of stochastic quantisation [5]. Quantitatively, the latter leads to the same infrared exponents as the Dyson-Schwinger equations, which reflects an equivalence between the two formulations under certain conditions [5, 6]. Interesting in their own right, such interrelations are especially useful for finding ways to go beyond the particular approximations employed in each case.

Here we present an investigation of the infrared sector of Landau gauge QCD by means of a nonperturbative flow equation [7, 8]. Central to the approach is the ef-

fective action  $\Gamma_k$ , where quantum fluctuations with momenta  $p^2 > k^2$  are already integrated out, and  $k$  denotes an infrared cut-off scale. The variation with  $k$  leads to a flow equation for  $\Gamma_k$ , thereby interpolating between the classical action in the ultraviolet and the full quantum effective action in the infrared where the cut-off is removed. Moreover, it has been shown that truncations can be systematically improved by means of an optimisation procedure [9]. The formalism has its particular merits in gauge theories as it is amiable to approximations being non-local in momenta and fields, precisely the behaviour we expect in QCD. So far it has been applied to Landau gauge QCD for a determination of the heavy quark effective potential [10] and effective quark interactions above the confinement scale [11]; for a brief review and further applications in Yang-Mills theories see [12] and [13] respectively.

Conceptually, the flow comprises a system of coupled differential equations for all Green functions which only involves dressed vertices and propagators. Hence, the correct RG-scaling is displayed by the full flow and truncations with the correct symmetry properties. This is one of the main contradistinctions to Dyson-Schwinger equations or stochastic quantisation which involve both bare and dressed Green functions. This makes it often difficult in the latter two approaches to implement the correct RG-scaling behaviour within truncations.

In this letter we explain how the full infrared momentum behaviour of Green functions is extracted within a fixed point regime. In Landau gauge QCD, this allows for a simple determination of infrared coefficients for gluon and ghost propagator as well as the running coupling. Results for general regulators are presented and interpreted in the context of an optimisation of the flow. The systematic extension of the truncation is discussed.

We begin by introducing the classical gauge fixed action for QCD in a general covariant gauge in four dimensions. Including also the ghost action, it is given by

$$S_{\text{cl}} = \frac{1}{2} \int \text{tr} F^2 + \frac{1}{2\xi} \int (D_\mu A^\mu)^2 + \int \bar{C} \cdot \partial_\mu D_\mu \cdot C. \quad (1)$$

The flow equation approach relies on a momentum cut-

off for the propagating degrees of freedom. Here, the momentum regularisation for both, the gluon and the ghost, is achieved by  $S_{\text{cl}} \rightarrow S_{\text{cl}} + \Delta S_k$  with

$$\Delta S_k = \frac{1}{2} \int A_a^\mu R_{\mu\nu}^{ab} A_b^\nu + \int \bar{C}_a R^{ab} C_b. \quad (2)$$

The functions  $R$  implement the infrared momentum cut-off at the momentum scale  $p^2 \approx k^2$ . The scale-dependence of the regularisation in (2) induces a flow equation for the effective action. With  $\phi = (A, C, \bar{C})$  and  $t = \ln k$ , the flow equation takes the explicit form

$$\begin{aligned} \partial_t \Gamma_k[\phi] = & \frac{1}{2} \int d^4 p G_{ab}^{\mu\nu}[\phi](p, p) \partial_t R_{\mu\nu}^{ba}(p) \\ & - \int d^4 p G^{ab}[\phi](p, p) \partial_t R^{ba}(p), \end{aligned} \quad (3)$$

where  $G[\phi](p, q) = (\Gamma_k^{(2)} + R)^{-1}(p, q)$  denotes the full regularised propagator. The flow is finite in both, the infrared and the ultraviolet, by construction. Effectively, the momentum integration in (3) only receives contributions for momenta in the vicinity of  $p^2 \approx k^2$ . Consequently it has a remarkable numerical stability. The flow solely depends on dressed vertices and propagators leading to consistent RG-scaling on either side of (3). The flow equations for propagators and vertices are obtained from (3) by functional derivatives.

To verify the Kugo-Ojima confinement criterion we have to determine the momentum behaviour of the propagators in the deep infrared regime

$$k^2 \ll p^2 \ll \Lambda_{\text{QCD}}^2. \quad (4)$$

In this domain, quantum fluctuations are already integrated out and the *physical* quantities as well as general vertex functions will show no  $k$  dependence. Moreover, (4) entails that we are already deep in the infrared regime. No new physics arises and the flow equation (3) only constitutes the shifting of the transition regime at momenta about  $k^2$  towards smaller scales. Accordingly, Green functions  $\Gamma_k^{(n)} \equiv \delta^{(n)} \Gamma_k / \delta \phi_1 \cdots \delta \phi_n$  can be parameterised as

$$\Gamma_k^{(n)}(p_i^2 \ll \Lambda_{\text{QCD}}^2) = z_n \hat{\Gamma}^{(n)}(p_i, p_i^2/k^2), \quad (5)$$

where  $p_i$  stands for the momenta  $p_1, \dots, p_n$ . Here  $z_n$  are possibly  $k$ -dependent prefactors accounting for an RG-scaling that leaves the full action invariant. This reduces the remaining  $k$ -dependence in  $\hat{\Gamma}^{(n)}$  to dimensionless ratios  $x = p^2/k^2$ . Hence, the interpolation between the physical infrared regime (4) and the regularised behaviour for  $p^2 \ll k^2$  is  $k$ -independent. This implies a fixed point behaviour. So far, fixed point investigations within the exact renormalisation group have been put forward within derivative expansions. Here we provide a new truncation scheme for accessing the full momentum dependence of Green functions in the infrared limit.

Taking advantage of their simplified structure in the physical regime (4), we introduce the following

parametrisation for the gluon and ghost two point function

$$\begin{aligned} \Gamma_{k,A}^{(2)}(p^2) &= z_A \cdot Z_A(x) \cdot p^2 \cdot \Pi(p) \cdot \mathbb{1} + \text{longitudinal}, \\ \Gamma_{k,C}^{(2)}(p^2) &= z_C \cdot Z_C(x) \cdot p^2 \cdot \mathbb{1}, \end{aligned} \quad (6)$$

where

$$Z(x) = x^\kappa (1 + \delta Z(x)).$$

Here  $\Pi_{\mu\nu}(p) = \delta_{\mu\nu} - p_\mu p_\nu / p^2$ ,  $\mathbb{1}_{ab} = \delta_{ab}$  and  $Z, \delta Z, \kappa$  without subscripts stand for  $Z_{A/C}, \delta Z_{A/C}, \kappa_{A/C}$ . The longitudinal part of the gluon two point function in (6) is not given here as it does not contribute to the Landau gauge flows considered below. The momentum dependence of (6) in the asymptotic regimes  $x \gg 1$  and  $x \ll 1$  fixes the functions  $\delta Z$  in both regimes: the prefactor  $x^\kappa$  encodes the leading behaviour for the momentum regime (4) where  $1 \ll x \ll \Lambda_{\text{QCD}}^2/k^2$ , hence  $\delta Z$  tends to zero. The  $k$ -independence of (6) at  $k = 0$  implies  $\partial_t \kappa = 0$  and  $\partial_t z = 2\kappa z$ . In the limit of small  $x$ , we recover the trivial momentum behaviour  $\Gamma_k^{(2)}(p^2) = m_k^2 + z_1 p^2 + O(p^3)$  as a consequence of the cut-off. Accordingly the functions  $\delta Z$  have the limits

$$\begin{aligned} \delta Z(x \rightarrow \infty) &\rightarrow 0, \\ \delta Z(x \rightarrow 0) &\rightarrow -1 + \mu x^{-(1+\kappa)} + O(x^{-\kappa}). \end{aligned} \quad (7)$$

We emphasise that the limit  $x \rightarrow \infty$  implies  $k \rightarrow 0$  as we demand  $x \ll \Lambda_{\text{QCD}}^2/k^2$ . In the intermediate region  $x \approx 1$  the propagators show a regulator-dependent interpolation between the physical infrared behaviour and the trivial cut-off regime. Also

$$\partial_t \delta Z(x) = -2x \partial_x \delta Z \quad (8)$$

reflects that the dependence of  $\delta Z$  on  $k$  stems from the interpolation property while its shape is independent of it. The vertices  $\Gamma_k^{(n)}$  in the present truncation are schematically given by

$$\Gamma_k^{(n)} = z_n S_{\text{cl}}^{(n)} \quad (9)$$

for  $n \geq 3$ . In the above parametrisation, the non-renormalisation of the ghost-gluon vertex at  $k = 0$  leads to

$$\kappa_A = -2\kappa_C, \quad \alpha_s = \frac{g^2}{4\pi} \frac{1}{z_A z_C^2}. \quad (10)$$

It follows from (10) and  $\partial_t z = 2\kappa z$  that the strong coupling constant achieves a fixed point in the infrared limit,  $\partial_t \alpha_s = 0$ . In particular, the  $z$ -factors for the three- and four-gluon vertices are proportional to  $\alpha_s^{1/2} z_A^{3/2}$  and  $\alpha_s z_A^2$ , respectively. Moreover it follows within the truncation (9) that no mass term for the ghost is present corresponding to  $\mu_C = 0$  in (7). In summary, the truncation (6),(9) with the properties (7),(10) satisfy the RG properties of Landau gauge QCD as well as the truncated Slavnov-Taylor identities valid in the presence of the regulator.

We proceed with determining the strong coupling fixed point  $\alpha_s$  and the infrared exponents  $\kappa$  from (3) in the truncations put down in (6). We fix the tensor structure of the regulators as  $R_{\mu\nu}^{ab} = R_A \delta_{\mu\nu} \delta^{ab}$  and  $R^{ab} = R_C \delta^{ab}$ , leaving the scalar part  $R = z p^2 r(x)$  at our disposal. The integrated flow leads to an integral equation for  $\delta Z$ ,

$$\delta Z_{A/C}(x) = \frac{\alpha_s}{\pi^2} N \int_x^\infty \frac{dx'}{x'^{2+\kappa}} f_{A/C}(x'), \quad (11)$$

where

$$\begin{aligned} f_A(x) &= -\frac{1}{3} \int_0^\infty dy y \int_{-1}^1 dt \frac{y(1-t^2)^{3/2} \mathcal{T}_C(y)}{u r_C(u) + Z_C(u)}, \\ f_C(x) &= \frac{x}{2} \int_0^\infty dy y \int_{-1}^1 dt (1-t^2)^{3/2} \\ &\quad \times \frac{1}{u} \left( \frac{y}{u r_A(u) + Z_A(u)} \mathcal{T}_C(y) + \frac{\mathcal{T}_A(y)}{r_C(u) + Z_C(u)} \right). \end{aligned}$$

and  $u = x + y + 2t\sqrt{xy}$ . We have also introduced the abbreviation  $2\mathcal{T} = k^2 G(\partial_t R)G$ , which reads explicitly

$$\mathcal{T}(y) = \frac{1}{y} \frac{\kappa r(y) - y r'(y)}{(r(y) + Z(y))^2}.$$

In (11), the integration over  $k$  was traded for an integration over  $x$ , using  $z = \hat{z} x^{-\kappa} (\Lambda_{\text{QCD}}^2/p^2)^\kappa$  with  $\partial_t \hat{z} = 0$ . The purely gluonic diagrams in (3) are suppressed by positive powers of  $p^2/\Lambda_{\text{QCD}}^2$  in the fixed point regime and do not contribute to (11).

For the sharp cut-off the kernel  $\mathcal{T}(x)$  simplifies and becomes  $\mathcal{T}(x) = \delta(x-1)/(zZ(1))$ . Then the  $t$  and  $y$  integrals in (11) can be performed analytically leading to a lengthy sum of hyper-geometric functions. For generic smooth cut-off,  $\kappa$  and  $\alpha_s$  are computed from (11) in the limit  $x \rightarrow 0$  as follows. On the right hand side of (11), the integrands  $f_{A/C}$  are suppressed for small momenta  $x$  and  $y$  due to  $\mathcal{T}_{A/C}$ . This implies that  $\delta Z$  can only contribute for sufficiently large arguments where it tends to zero due to (7). Therefore we can safely neglect  $\delta Z$  under the momentum trace on the right hand side of (11) in a first step. Then, we proceed to fully consistent solutions  $\delta Z$  of (11) by iteration. On the left hand side of the gluon equation the term  $\delta Z_A(x \rightarrow 0)$  is determined by (7). In the ghost equation, we have to carefully identify the terms. For  $\kappa_C > 0$  we read-off from (7) that  $\delta Z_C(x \rightarrow 0)$  diverges. However,  $\Gamma_{k,C}^{(2)}(p^2) = z_C x^{\kappa_C} (1 + \delta Z_C(x))$  has a Taylor expansion about  $x = 0$  due to the IR regularisation. Hence the coefficient of the constant term in  $\delta Z_C$  is  $-1$ .

The integral equation (11) can be solved numerically for  $\kappa$  and  $\alpha_s$ . We have restricted ourselves to the domain  $\kappa_C \in (\frac{1}{2}, 1)$  [14]. For *general* regulator  $r(x)$  we obtain

$$\kappa_C = 0.59535\dots, \quad \alpha_s = 2.9717\dots, \quad (12)$$

which agrees with the analytic DS result [6]. This might seem surprising given the completely different representations. However, the equivalence can be proven as follows:

in the derivation of (12) we employed the truncation (6) and (9) with the additional constraint  $\delta Z \approx 0$  under the momentum trace in (3). Integrating the flow (3) in this setting from 0 to  $k$  leads to

$$\begin{aligned} z_A p^2 x^{\kappa_A} \delta Z_A &= \text{Tr} [G S^{(3)} G S^{(3)} - \tfrac{1}{2} G S^{(4)} G]_0^k, \\ z_C p^2 x^{\kappa_C} \delta Z_C &= \text{Tr} [G S^{(3)} G S^{(3)}]_0^k, \end{aligned} \quad (13)$$

for the gluon and ghost two point function. Eq. (13) already implies an UV renormalisation at  $p^2 = k^2$ . This is reflected by (11) being finite. Within the present truncation the renormalised DS equations used in [6] are equivalent to (13). Beyond this truncation the DS equation and the flow equation differ. Apart from additional  $t$ -derivative terms, all vertices in (13) get dressed as distinguished from the DS equation, where both bare and dressed vertices are present. Consequently, the flow equation (3) carries the correct RG properties if the truncation for  $\Gamma_k$  does. In particular, equation (13) with the truncation (6) and (9) allows for a fully self-consistent renormalisation. Both the left and right hand sides of (13) transform the same way under RG scalings, including the purely gluonic terms dropped in (11). Indeed, it is the latter fact which makes their neglect fully consistent. The RG properties are most obvious for regulators proportional to the wave function renormalisations  $z, R \simeq z \hat{R}$ . Then, the propagators can be written as  $G = z^{-1} G_0$  where  $G_0$  does not involve further wave function renormalisation factors. Also, the vertex functions involve the appropriate multiplicative powers of  $z$  following from (10) and (9).

We proceed by iterating the above solution, also taking into account  $\delta Z \neq 0$  on the right hand side of the flow. Then (11) can be used for a numerical iterative determination of  $\delta Z$  and  $\kappa, \alpha_s$ . We have computed  $\alpha_s$  and  $\kappa$  for various classes of regulators. The exponent  $\kappa_C$  takes values in an interval bounded by the values (12) from above and the iterated sharp cut-off result  $\kappa_C \simeq 0.539$  from below. However, the values (12) are singled out by their physical relevance:  $\kappa$  and  $\alpha_s$  depend on the regulator. This reflects the fact that the truncation (6),(9) and the choice of  $R$  define the approximation to the full problem. We expect that global extrema of  $\kappa(R), \alpha_s(R)$  represent an optimised approximation based on (6),(9), see [9]. In general it is very difficult to translate this idea into properties of optimal regulators or even in statements about the values of optimal observables. Here we present a new idea how to extract this information: let us assume that we have identified a regulator  $R_0$  which is an extremum in the space of regulators in the sense that

$$\left[ \text{Tr} \left( \delta R \frac{\delta}{\delta R} \right) \Gamma_k^{(n)} \right]_{R=R_0} = 0 \quad (14)$$

for all  $\delta R$ . Eq. (14) implies that  $\kappa(R_0), \alpha_s(R_0)$  are extrema, reflecting an optimisation of physical quantities. Note that the  $z_n(R_0)$  by themselves are not physical quantities but they can be extrema up to RG scalings. However, (14) is more restrictive. For  $n = 2$ ,

(14) is the additional demand, that  $\delta Z(x)$  is an extremal curve. Hence the question arises whether solutions  $R_0$  to (14) exist at all: the fixed point assumption underlying the whole analysis is valid for regulators  $R$  that only lead to a suppression of infrared modes. This implies a monotonous interpolation between the physical infrared regime  $k^2 \ll p^2 \ll \Lambda_{\text{QCD}}^2$  and the infrared regularised regime  $p^2 \ll k^2$ . Then the extremal curves exist and are given by vanishing  $\delta Z$  [15].

The above considerations allow us to identify the optimal values for  $\kappa_C(R)$  and  $\alpha_s(R)$  without further computation. The correction terms proportional to  $\delta Z$  under the momentum trace are negligible near optimal regulators and we are left with the non-iterated equation (11) with an optimal regulator  $R_0$ . However, we have already shown that the non-iterated solutions are independent of the regulator and we conclude that  $\kappa_C(R_0), \alpha_s(R_0)$  are given by (12). We are also in a position to devise optimal regulators. For these regulators the effects of  $\delta Z$  on the right hand side of (11) have to disappear. They need not to satisfy (14) which only applies to a small sub-set of optimal regulators. In Fig. 1 we show  $\delta Z_{A/C}$  for the class of regulators

$$r_C(x) = z_C \theta^{-1}(1-x), \quad r_A(x) = \gamma \frac{z_A}{x(1+x)}. \quad (15)$$

The  $\gamma$ -dependence of  $\delta Z_A$  is negligible on the scale of Fig. 1. The ghost regulator in (15) is a sharp cut-off while the gluon regulator is a rather soft mass-like cut-off. In the limit  $\gamma \rightarrow \infty$  the solution of (11) approaches the optimal values (12). One verifies that all contributions from  $\delta Z$  to the right hand side of (11) tend to zero for  $\gamma \rightarrow \infty$  and we are left with the non-iterated result (12). Hence (15) with  $\gamma \rightarrow \infty$  belongs to the class of optimal regulators. It is not in the extremal subset as only the ghost propagator satisfies (14) with  $\delta Z_C(x \neq 0) = 0$ .

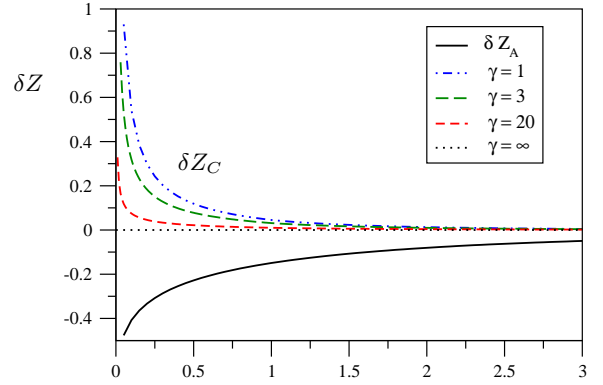


Figure 1:  $\delta Z$  for regulators with (15)

In summary, we have shown how the non-trivial momentum dependence of Green functions in the infrared sector is extracted within the exact renormalisation group. To that end we have developed new fixed point and optimisation techniques applicable to this situation. This reasoning has been applied to the infrared behaviour of ghost and gluon propagators in Landau gauge QCD. Our results provide further evidence for the Kugo-Ojima scenario of confinement: the ghost propagator is infrared enhanced while the gluon propagator develops a mass gap. In a truncation with dressed propagators, we have recovered previous results from Dyson-Schwinger equations in a very simple manner. This is noteworthy also in view of the qualitative differences between these approaches. Our method allows for a straightforward improvement of the truncation by adding vertex corrections and further terms with the correct RG properties. The same applies to dynamical quarks. In both extensions the correct RG scaling as well as the inherent finiteness of the integrated flow are most important. Further results and details will be presented elsewhere.

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  - [14] The  $\kappa, \alpha_s$  run from their perturbative values into the regime  $\kappa_C \in (\frac{1}{2}, 1)$ . We are interested in a more refined analysis of their infrared values rather than in the flow.
  - [15] This is valid up to RG scalings.